

# On Paraconsistent Weakening of Intuitionistic Negation

Zoran Majkić

International Society for Research in Science and Technology  
PO Box 2464 Tallahassee, FL 32316 - 2464 USA  
majk.1234@yahoo.com

**Abstract.** In [1], systems of weakening of intuitionistic negation logic called  $Z_n$  and  $CZ_n$  were developed in the spirit of da Costa's approach(c.f. [2]) by preserving, differently from da Costa, its fundamental properties: antitonicity, inversion and additivity for distributive lattices. However, according to [3], those systems turned out to be not paraconsistent but extensions of intuitionistic logic. Taking into account of this result, we shall here make some observations on the modified systems of  $Z_n$  and  $CZ_n$ , that are paraconsistent as well.

**Keywords:** Paraconsistent logic, Intuitionistic logic, Majkić's systems  $Z_n$  and  $CZ_n$ .

## 1 Introduction

The big challenge for paraconsistent logics is to avoid allowing contradictory theories to explode and derive anything else and still to reserve a respectable logic, that is, a logic capable of drawing reasonable conclusions from contradictory theories.

There are different approaches to paraconsistent logics: The first is the non constructive approach, based on abstract logic (as LFI [4]), where logic connectives and their particular semantics are not considered. The second is the constructive approach and is divided in two parts: axiomatic proof theoretic (cases of da Costa [2] and [5,6,7]), and many-valued (case [8]) model theoretic based on truth-functional valuations (that is, it satisfies the truth-compositionality principle). The best case is when we obtain both proof and model theoretic definition which are mutually sound and complete.

One of the main founders with Stanislaw Jaskowski [9], da Costa, built his propositional paraconsistent system  $C_\omega$  in [2] by weakening the logic negation operator  $\neg$ , in order to avoid the explosive inconsistency [4,10] of the classic propositional logic, where the ex falso quodlibet proof rule  $\frac{A, \neg A}{B}$  is valid. In fact, in order to avoid this classic logic rule, he changed the semantics for the negation operator, so that:

- NdC1: in these calculi the principle of non-contradiction, in the form  $\neg(A \wedge \neg A)$ , should not be a generally valid schema, but if it does hold for formula  $A$ , it is a well-behaved formula, and is denoted by  $A^\circ$ ;
- NdC2: from two contradictory formulae,  $A$  and  $\neg A$ , it would not in general be possible to deduce an arbitrary formula  $B$ . That is it does not hold the falso quodlibet proof rule  $\frac{A, \neg A}{B}$ ;
- NdC3: it should be simple to extend these calculi to corresponding predicate calculi (with or without equality);

- NdC4: they should contain most parts of the schemata and rules of classical propositional calculus which do not interfere with the first conditions

In fact Da Costa's paraconsistent propositional logic is made up of the unique Modus Ponens inferential rule (MP),  $A, A \Rightarrow B \vdash B$ , and two axiom subsets. But before stating them we need the following definition as it is done in da Costa's systems (c.f. [2, p.500]), which uses three binary connectives,  $\wedge$  for conjunction,  $\vee$  for disjunction and  $\Rightarrow$  for implication:

**Definition 1.** Let  $A$  be a formula and  $1 \leq n < \omega$ . Then, we define  $A^\circ, A^n, A^{(n)}$  as follows:

$$A^\circ =_{\text{def}} \neg(A \wedge \neg A), \quad A^n =_{\text{def}} A^{\overbrace{\circ \circ \dots \circ}^n}, \quad \text{and} \quad A^{(n)} =_{\text{def}} A^1 \wedge A^2 \wedge \dots \wedge A^n.$$

The first one is for the positive propositional logic (without negation), composed by the following eight axioms, borrowed from the classic propositional logic of the Kleene  $L_4$  system, and also from the more general propositional *intuitionistic* system (these two systems differ only regarding axioms with the negation operator),

(IPC<sup>+</sup>) POSITIVE LOGIC AXIOMS:

- (1)  $A \Rightarrow (B \Rightarrow A)$
- (2)  $(A \Rightarrow B) \Rightarrow ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow C))$
- (3)  $A \Rightarrow (B \Rightarrow (A \wedge B))$
- (4)  $(A \wedge B) \Rightarrow A$
- (5)  $(A \wedge B) \Rightarrow B$
- (6)  $A \Rightarrow (A \vee B)$
- (7)  $B \Rightarrow (A \vee B)$
- (8)  $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \vee B) \Rightarrow C))$

and change the original axioms for negation of the classic propositional logic, by defining semantics of negation by the following subset of axioms:

(NLA) LOGIC AXIOMS FOR NEGATION:

- (9)  $A \vee \neg A$
- (10)  $\neg\neg A \Rightarrow A$
- (11)  $B^{(n)} \Rightarrow ((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))$  (Reductio relativization axiom)
- (12)  $(A^{(n)} \wedge B^{(n)}) \Rightarrow ((A \wedge B)^{(n)} \wedge (A \vee B)^{(n)} \wedge (A \Rightarrow B)^{(n)})$

□

It is easy to see that the axiom (11) relativizes the classic *reductio* axiom  $(A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A)$  (which is equivalent to the contraposition axiom  $(A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A)$  and the trivialization axiom  $\neg(A \Rightarrow A) \Rightarrow B$ ), *only* for propositions  $B$  such that  $B^{(n)}$  is valid, and in this way avoids the validity of the classic ex falso quodlibet proof rule. It provides a qualified form of reductio, helping to prevent general validity of  $B^{(n)}$  in the paraconsistent logic  $C_n$ . The axiom (12) regulates only the propagation of n-consistency. It is easy to verify that n-consistency also propagates through negation, that is,  $A^{(n)} \Rightarrow (\neg A)^{(n)}$  is provable in  $C_n$ . So that for any fixed  $n$  (from 0 to  $\omega$ ) we obtain a particular da Costa paraconsistent logic  $C_n$ .

One may regard  $C_\omega$  as a kind of syntactic limit [11] of the calculi in the hierarchy.

Each  $C_n$  is strictly weaker than any of its predecessors, i.e., denoting by  $Th(S)$  the set of theorems of calculus  $S$ , we have:

$$Th(CPL) \supset Th(C_1) \supset \dots \supset Th(C_n) \supset \dots \supset Th(C_\omega).$$

Thus we are fundamentally interested in the  $C_1$  system which is a paraconsistent logic closer to the CPL (Classic propositional logic), that is,  $C_1$  is the paraconsistent logic of da Costa's hierarchy obtained by minimal change of CPL.

It is well known that the classic propositional logic based on the classic 2-valued complete distributive lattice  $(\mathbf{2}, \leq)$  with the set  $\mathbf{2} = \{0, 1\}$  of truth values, has a truth-compositional model theoretic semantics. For this da Costa calculi is not given any truth-compositional model theoretic semantics instead.

The *non-truth-functional* bivaluations (mappings from the set of well-formed formulae of  $C_n$  into the set  $\mathbf{2}$ ) used in [12,13] induce the decision procedure for  $C_n$  known as quasi-matrices instead. In this method, a negated formulae within truth-tables must branch: if  $A$  takes the value 0 then  $\neg A$  takes the value 1 (as usual), but if  $A$  takes the value 1 then  $\neg A$  can take either the value 0 or the value 1; both possibilities must be considered, as well as the other axioms governing the bivaluations.

Consequently, the da Costa system still needs a kind of (relative) compositional model-theoretic semantics. Based on these observations, in [1] are explained some weak properties of Da Costa weakening for a negation operator, and was shown that it is not antitonic, differently from the negations in the classic and intuitionistic propositional logics (that have the truth-compositional model theoretic semantics). The axioms for negation in CPL are as follows:

(NCLA) CLASSIC AXIOMS FOR NEGATION:

- (9)  $A \vee \neg A$
- (10c)  $(A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A)$
- (11c)  $A \Rightarrow (\neg A \Rightarrow B)$
- (12c)  $0 \Rightarrow A, A \Rightarrow 1$

while for the intuitionistic logic we eliminate the axiom (9).

The negation in the classic and intuitionistic logics are not paraconsistent (see for example Proposition 30, pp 118, in [8]), so that Majkić's idea in [1] was to make a weakening of the intuitionistic negation by considering only its general antitonic property: in fact the formula  $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$  is a thesis in both classic and intuitionistic logics. Consequently, his idea was to make da Costa weakening of the *intuitionistic* negation [1], that is, to define the system  $Z_n$  for each  $n$  by adding the following axioms to the system  $IPC^+$ :

- (11)  $B^{(n)} \Rightarrow ((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))$
- (12)  $(A^{(n)} \wedge B^{(n)}) \Rightarrow ((A \wedge B)^{(n)} \wedge (A \vee B)^{(n)} \wedge (A \Rightarrow B)^{(n)})$
- (9b)  $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$
- (10b)  $1 \Rightarrow \neg 0, \neg 1 \Rightarrow 0$
- (11b)  $A \Rightarrow 1, 0 \Rightarrow A$
- (12b)  $(\neg A \wedge \neg B) \Rightarrow \neg(A \vee B)$

Finally, the hierarchy  $CZ_n$  is obtained by adding the following axiom:

- (13b)  $\neg(A \wedge B) \Rightarrow (\neg A \vee \neg B)$

The result provided in [3] is that in the above formulation of the system  $Z_n$ , axioms (11), (12) and (12b) are redundant in the sense that those formulas can be derived

from the other axioms (9b), (10b) and (11b) in addition to  $\text{IPC}^+$ . Obviously, the formulation of  $\text{CZ}_n$  is given by adding the axiom (13b). As a result, systems  $Z_n$  and  $\text{CZ}_n$  do not form a hierarchy but are single systems. It is also proved that formulas ‘ $(A \Rightarrow (A \wedge \neg A)) \Rightarrow \neg A$ ’ and ‘ $A \Rightarrow (\neg A \Rightarrow B)$ ’ can be proved in  $Z_n$  which shows that  $Z_n$  and  $\text{CZ}_n$  are extensions of intuitionistic propositional calculus and therefore not paraconsistent.

In fact, the introduction of the axiom  $\neg 1 \Rightarrow 0$  in the system  $Z_n$  is *not necessary* for the all obtained results in [1]: this formula was responsible for the fact that  $Z_n$  is not paraconsistent.

In what follows we will present the properties of this modified system, by eliminating this formulae from the system  $Z_n$ .

## 2 Paraconsistent weakening of negation

In what follows we consider modified systems of  $Z_n$  and  $\text{CZ}_n$  which can be obtained by eliminating the formula ‘ $\neg 1 \Rightarrow 0$ ’ of axiom (10b) from the systems  $Z_n$  and  $\text{CZ}_n$ . Notice that this axiom is not necessary in order to have additive modal negation operator that can be modeled by Birkhoff’s polarity as required in [1]. We shall refer to these systems as  $mZ_n$  and  $m\text{CZ}_n$  respectively and also refer to the modified axiom as (10b)’.

Thus, all results obtained in [1] are preserved for this logic: what we need is only to eliminate the sequent  $\neg 1 \vdash 0$  from (5a) in Definition 7 (Gentzen-like system) in [1] as well.

Consequently, these modified systems  $mZ_n$  and  $m\text{CZ}_n$  have the Kripke possible world semantics for these two paraconsistent logics (defined by Definition 6 in [1]), and based on it, the many-valued semantics based on functional hereditary distributive lattice of algebraic truth-values. Finally, this many-valued (and Kripke) semantics, based on model-theoretic entailment, is adequate, that is, sound and complete w.r.t. the proof-theoretic da Costa axiomatic systems of these two paraconsistent logics  $mZ_n$  and  $m\text{CZ}_n$ .

We now prove some another results on  $mZ_n$  and  $m\text{CZ}_n$ :

**Proposition 1** *Following formulas are derivable in  $mZ_n$  (we denote by  $A \equiv B$  the formulae  $(A \Rightarrow B) \wedge (B \Rightarrow A)$ ):*

$$((A \Rightarrow B) \wedge (A \Rightarrow C)) \Rightarrow (A \Rightarrow (B \wedge C)) \quad (\text{T0})$$

$$(A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C)) \quad (\text{T1})$$

$$(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)) \quad (\text{T2})$$

$$(A \Rightarrow (B \Rightarrow C)) \equiv ((A \wedge B) \Rightarrow C) \quad (\text{T3})$$

This is obvious since  $mZ_n$  contains  $\text{IPC}^+$ .

**Theorem 1** *Systems  $mZ_n$  and  $m\text{CZ}_n$  are paraconsistent.*

**Proof:** Just interpret the negation as a function always giving truth value 1 whereas other connectives interpreted in a standard way done in two valued for classical propositional

calculus.

□

It should be noted that even though we have the above theorem, the following formula  $(A \wedge \neg A) \Rightarrow \neg B$  is still derivable, as we can show by the following lemma:

**Lemma 1.** *The following formulae are derivable in  $mZ_n$ :*

$$\begin{aligned} (A \wedge \neg A) &\Rightarrow \neg B && \text{NEFQ} \\ \neg\neg(A \wedge B) &\Rightarrow (\neg\neg A \wedge \neg\neg B) && (\heartsuit) \\ \neg((A * B)^n) &\Rightarrow (\neg(A^n) \vee \neg(B^n)) && (\clubsuit) \end{aligned}$$

where  $*$   $\in \{\Rightarrow, \wedge, \vee\}$ .

**Proof:** Let us derive NEFQ:

$$\begin{aligned} 1 \quad & A \Rightarrow (B \Rightarrow A) && [(1)] \\ 2 \quad & (B \Rightarrow A) \Rightarrow (\neg A \Rightarrow \neg B) && [(9b)] \\ 3 \quad & A \Rightarrow (\neg A \Rightarrow \neg B) && [1, 2, (T2)] \\ 4 \quad & (A \wedge \neg A) \Rightarrow \neg B && [3, (T3), (MP)] \end{aligned}$$

Notice that (NEFQ) is not desirable for some paraconsistent

Let us derive  $\heartsuit$  now. We will only prove the following, since the case in which  $\neg\neg A$  is replaced by  $\neg\neg B$  can be proved analogously:

$$\neg\neg(A \wedge B) \Rightarrow \neg\neg A$$

This can be proved easily by making use of axioms (3) and (9b).

Let us derive  $\clubsuit$  now. The proof runs as follows:

$$\begin{aligned} 1 \quad & \neg((A * B)^n) \equiv \neg\neg((A * B)^{n-1} \wedge \neg(A * B)^{n-1}) && [\text{Definition of } A^n] \\ 2 \quad & \neg\neg((A * B)^{n-1} \wedge \neg(A * B)^{n-1}) \Rightarrow (\neg\neg(A * B)^{n-1} \wedge \neg\neg\neg(A * B)^{n-1}) && [(\heartsuit)] \\ 3 \quad & (\neg\neg(A * B)^{n-1} \wedge \neg\neg\neg(A * B)^{n-1}) \Rightarrow \neg(A^n) && [(NEFQ)] \\ 4 \quad & \neg(A^n) \Rightarrow (\neg(A^n) \vee \neg(B^n)) && [(6)] \\ 5 \quad & \neg((A * B)^n) \Rightarrow (\neg(A^n) \vee \neg(B^n)) && [1, 2, 3, 4, (T2), (MP)] \end{aligned}$$

This completes the proof.

□

Let us show now that the axioms (11) and (12) are redundant in the System  $mZ_n$ .

**Theorem 2** *The axioms (11) and (12) are redundant in  $mZ_n$  in the sense that they can be proved by another axioms.*

**Proof:** The redundance of the axiom (11) can be proved as follows:

$$\begin{aligned} 1 \quad & (A \Rightarrow (B \wedge \neg B)) \Rightarrow (\neg(B \wedge \neg B) \Rightarrow \neg A) && [(9b)] \\ 2 \quad & \neg(B \wedge \neg B) \Rightarrow ((A \Rightarrow (B \wedge \neg B)) \Rightarrow \neg A) && [1, (T1), (MP)] \\ 3 \quad & B^{(n)} \Rightarrow B^1 && [\text{Definition of } B^{(n)}] \\ 4 \quad & B^{(n)} \Rightarrow \neg(B \wedge \neg B) && [\text{Definition of } B^1] \\ 5 \quad & B^{(n)} \Rightarrow ((A \Rightarrow (B \wedge \neg B)) \Rightarrow \neg A) && [2, 4, (T2), (MP)] \end{aligned}$$

Let us prove the redundance of the axiom (12). It would be sufficient to prove the following in order to prove the desired result:

$$(A^{(n)} \wedge B^{(n)}) \Rightarrow (A * B)^n \quad (\diamond)$$

Indeed, if we have  $(\diamond)$  at hand then we can prove

$$(A^{(n)} \wedge B^{(n)}) \Rightarrow (A * B)^m$$

for any  $1 \leq m \leq n$  and combining all these cases, we obtain

$$(A^{(n)} \wedge B^{(n)}) \Rightarrow (A * B)^{(n)}$$

which is axiom (12). So, we now prove  $(\diamond)$  which runs as follows:

$$\begin{array}{ll}
1 & (A^{(n)} \wedge B^{(n)}) \Rightarrow (((A * B)^{n-1} \wedge \neg(A * B)^{n-1}) \Rightarrow (\neg(A^{n-1}) \vee \neg(B^{n-1}))) \text{ [(♣)]} \\
2 & (A^{(n)} \wedge B^{(n)}) \Rightarrow (((A * B)^{n-1} \wedge \neg(A * B)^{n-1}) \Rightarrow ((A^{n-1} \wedge \neg(A^{n-1})) \vee \\
& (B^{n-1} \wedge \neg(B^{n-1})))) \\
& [1, \text{Definition1}] \\
3 & (((A * B)^{n-1} \wedge \neg(A * B)^{n-1}) \Rightarrow ((A^{n-1} \wedge \neg(A^{n-1})) \vee (B^{n-1} \wedge \neg(B^{n-1})))) \Rightarrow \\
& (\neg((A^{n-1} \wedge \neg(A^{n-1})) \vee (B^{n-1} \wedge \neg(B^{n-1})))) \Rightarrow \neg((A * B)^{n-1} \wedge \neg(A * B)^{n-1})) \\
& [(9b)] \\
4 & (A^{(n)} \wedge B^{(n)}) \Rightarrow (\neg((A^{n-1} \wedge \neg(A^{n-1})) \vee (B^{n-1} \wedge \neg(B^{n-1})))) \Rightarrow \neg((A * B)^{n-1} \wedge \neg(A * B)^{n-1})) \\
& [2, 3, (T2), (MP)] \\
5 & (\neg(A^{n-1} \wedge \neg(A^{n-1})) \wedge \neg(B^{n-1} \wedge \neg(B^{n-1}))) \Rightarrow \neg((A^{n-1} \wedge \neg(A^{n-1})) \vee \\
& (B^{n-1} \wedge \neg(B^{n-1}))) \text{ [(12b)]} \\
6 & (A^{(n)} \wedge B^{(n)}) \Rightarrow ((A^n \wedge B^n) \Rightarrow (A * B)^n) [4, (T1), 5, (T2), \text{Definition1}, (MP)] \\
7 & (A^{(n)} \wedge B^{(n)}) \Rightarrow (A^n \wedge B^n) \text{ [Definition1]} \\
8 & (A^{(n)} \wedge B^{(n)}) \Rightarrow (A * B)^n \text{ [6, 7, (2), (MP)]}
\end{array}$$

This completes the proof.

□

After all, we now know that systems  $mZ_n$  do not form a hierarchy but are equivalent to a single system which consists of  $IPC^+$  together with axioms (9b), (10b)', (11b), (12b) and  $mCZ_n$  can be formulated by adding (13b) to these formulas. Note also that we didn't make any use of axioms (10b)' and (11b) in proving Theorem 2.

Although it is not directly connected to the story of  $mZ_n$  and  $mCZ_n$ , it should be noted that propagation axiom for negation, i.e. the following formula can be derived in an analogous manner:

$$A^{(n)} \Rightarrow (\neg A)^{(n)}$$

Therefore, propagation axioms can be fully *proved* in systems  $mZ_n$  and  $mCZ_n$ .

### 3 Semantics of negation based on Bikhoff's polarity

In [1] (Proposition 3) was demonstrated that the positive fragment of these two systems corresponds to the distributive lattice  $(X, \leq)$  (positive fragment of the Heyting algebra), where the logic implication corresponds to the relative pseudocomplement, 0, 1 are bottom and top elements in  $X$  respectively.

Now we may introduce a hierarchy of negation operators [8] for many-valued logics based on complete lattices of truth values  $(X, \leq)$ , w.r.t their homomorphic properties: the negation with the lowest requirements (antitonic) denominated "general" negation can be defined in any complete lattice (see example in [1]):

**Definition 2.** HIERARCHY OF NEGATION OPERATORS: Let  $(X, \leq, \wedge, \vee)$  be a complete lattice. Then we define the following hierarchy of negation operators on it:

1. A *general negation* is a monotone mapping between posets ( $\leq^{OP}$  is inverse of  $\leq$ ),  $\neg : (X, \leq) \rightarrow (X, \leq)^{OP}$ , such that  $\{1\} \subseteq \{y = \neg x \mid x \in X\}$ .
2. A *split negation* is a general negation extended into join-semilattice homomorphism,  $\neg : (X, \leq, \vee, 0) \rightarrow (X, \leq, \vee, 0)^{OP}$ , with  $(X, \leq, \vee, 0)^{OP} = (X, \leq^{OP}, \vee^{OP}, 0^{OP})$ ,  $\vee^{OP} = \wedge$ ,  $0^{OP} = 1$ .
3. A *constructive negation* is a general negation extended into full lattice homomorphism,  $\neg : (X, \leq, \wedge, \vee) \rightarrow (X, \leq, \wedge, \vee)^{OP}$ , with  $(X, \leq, \wedge, \vee)^{OP} = (X, \leq^{OP}, \wedge^{OP}, \vee^{OP})$ , and  $\wedge^{OP} = \vee$ .
4. A *De Morgan negation* is a constructive negation when the lattice homomorphism is an involution ( $\neg\neg x = x$ ).

The names given to these different kinds of negations follow from the fact that a split negation introduces the second right adjoint negation, that a constructive negation satisfies the constructive requirement (as in Heyting algebras)  $\neg\neg x \geq x$ , while De Morgan negation satisfies well known De Morgan laws:

**Lemma 2.** NEGATION PROPERTIES): Let  $(X, \leq)$  be a complete lattice. Then the following properties for negation operators hold: for any  $x, y \in X$ ,

1. for general negation:  $\neg(x \vee y) \leq \neg x \wedge \neg y$ ,  $\neg(x \wedge y) \geq \neg x \vee \neg y$ , with  $\neg 0 = 1$ .
2. for split negation:  $\neg(x \vee y) = \neg x \wedge \neg y$ ,  $\neg(x \wedge y) \geq \neg x \vee \neg y$ . It is an additive modal operator with right adjoint (multiplicative) negation  $\sim : (X, \leq)^{OP} \rightarrow (X, \leq)$ , and Galois connection  $\neg x \leq^{OP} y$  iff  $x \leq \sim y$ , such that  $x \leq \sim \neg x$  and  $x \leq \neg \sim x$ .
3. for constructive negation:  $\neg(x \vee y) = \neg x \wedge \neg y$ ,  $\neg(x \wedge y) = \neg x \vee \neg y$ . It is a selfadjoint operator,  $\neg = \sim$ , with  $x \leq \neg\neg x$  satisfying proto De Morgan inequalities  $\neg(\neg x \vee \neg y) \geq x \wedge y$  and  $\neg(\neg x \wedge \neg y) \geq x \vee y$ .
4. for De Morgan negation ( $\neg\neg x = x$ ): it satisfies also De Morgan laws  $\neg(\neg x \vee \neg y) = x \wedge y$  and  $\neg(\neg x \wedge \neg y) = x \vee y$ , and is contrapositive, i.e.,  $x \leq y$  iff  $\neg x \geq \neg y$ .

Proof can be found in [8].

**Remark:** We can see (as demonstrated in [1] that the system  $mZ_n$  without axiom (12b) corresponds to a particular case of *general* negation, that the whole system  $mZ_n$  corresponds to a particular case of *split* negation, while the system  $mCZ_n$  corresponds to a particular case of *constructive* negation.

The Galois connections can be obtained from any binary relation based on a set  $\mathcal{W}$  [14] (Birkhoff *polarity*) in a canonical way:

If  $(\mathcal{W}, \mathcal{R})$  is a set with a particular relation based on a set  $\mathcal{W}$ ,  $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ , with mappings  $\lambda : \mathcal{P}(\mathcal{W}) \rightarrow \mathcal{P}(\mathcal{W})^{OP}$ ,  $\rho : \mathcal{P}(\mathcal{W})^{OP} \rightarrow \mathcal{P}(\mathcal{W})$ , such that for subsets  $U, V \in \mathcal{P}(\mathcal{W})$ ,

$\lambda U = \{w \in \mathcal{W} \mid \forall u \in U. ((u, w) \in \mathcal{R})\}$ ,  $\rho V = \{w \in \mathcal{W} \mid \forall v \in V. ((w, v) \in \mathcal{R})\}$ , where  $(\mathcal{P}(\mathcal{W}), \subseteq)$  is the *powerset poset* complete distributive lattice with bottom element empty set  $\emptyset$  and top element  $\mathcal{W}$ , and  $\mathcal{P}(\mathcal{W})^{OP}$  its dual (with  $\subseteq^{OP}$  inverse of  $\subseteq$ ), then we have the induced Galois connection  $\lambda \dashv \rho$ , i.e.,  $\lambda U \subseteq^{OP} V$  iff  $U \subseteq \rho V$ .

It is easy to verify that  $\lambda$  and  $\rho$  are two antitonic set-based operators which invert empty

set  $\emptyset$  into  $\mathcal{W}$ , thus can be used as set-based negation operators. The negation as modal operator has a long history [15].

We denote by  $\mathfrak{R}$  the class of such binary incompatibility relations  $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$  which are also hereditary, that is

if  $(u, w) \in \mathcal{R}$  and  $(u, w) \preceq (u', w')$  then  $(u', w') \in \mathcal{R}$ ,

where  $(u, w) \preceq (u', w')$  iff  $u \leq u'$  and  $w \leq w'$ .

Analogously to demonstration given in [1], it is easy to see that, for any given hereditary incompatibility relation  $\mathcal{R}$ , the additive algebraic operator  $\lambda$  can be used as the split negation for  $mZ_n$  (or constructive negation, when  $\lambda$  is selfadjoint, i.e.,  $\lambda = \rho$ , for  $mCZ_n$ ).

**Corollary 1** *Each split negation (modal negation), based on the hereditary incompatible relation of Birkhoff polarity, satisfies the Da Costa weakening axioms (11) and (12).*

**Proof:** for the Birkhoff polarity we have that for any  $U, V \subseteq \mathcal{W}$  holds the following additivity property,

$$\lambda(U \cup V) = \lambda U \bigcup^{OP} \lambda V = \lambda U \cap \lambda V, \text{ with } \lambda \emptyset = \emptyset^{OP} = \mathcal{W}.$$

It is well known that Heyting algebra operators are closed for hereditary subsets, so that  $\lambda$  applied to a hereditary subset  $U$  has to result in a hereditary subset  $\lambda(U)$  as well, and the Lemma 2 in [1] demonstrates that it is satisfied if the relation  $\mathcal{R}$  is hereditary.

It is enough now to prove that in  $mZ_n$  the following formulae are valid (the logic negation operator  $\neg$  corresponds to the algebraic operator  $\lambda$ ):

$$\neg(A \vee B) \equiv (\neg A \wedge \neg B), \text{ and } \neg 0 \equiv 1.$$

Indeed, we can derive this as follows: Indeed, we can derive this as follows:

- 1  $(1 \Rightarrow \neg 0) \Rightarrow ((\neg 0 \Rightarrow 1) \Rightarrow ((0 \Rightarrow \neg 1) \wedge (\neg 1 \Rightarrow 0)))$  [(3)]
  - 2  $(\neg 0 \Rightarrow 1) \Rightarrow ((0 \Rightarrow \neg 1) \wedge (\neg 1 \Rightarrow 0))$  [1, (10b), (MP)]
  - 3  $(0 \Rightarrow \neg 1) \wedge (\neg 1 \Rightarrow 0)$  [2, (11b), (MP)]
  - 4  $\neg 0 \equiv 1$  [3, by def. of  $\equiv$ ],
- and,
- 1  $(A \Rightarrow (A \vee B)) \Rightarrow (\neg(A \vee B) \Rightarrow \neg A)$  [(9b)]
  - 2  $(B \Rightarrow (A \vee B)) \Rightarrow (\neg(A \vee B) \Rightarrow \neg B)$  [(9b)]
  - 3  $\neg(A \vee B) \Rightarrow \neg A$  [1, (6), (MP)]
  - 4  $\neg(A \vee B) \Rightarrow \neg B$  [2, (7), (MP)]
  - 5  $(\neg(A \vee B) \Rightarrow \neg A) \wedge (\neg(A \vee B) \Rightarrow \neg B)$  [3, 4, (3), (MP)]
  - 6  $((\neg(A \vee B) \Rightarrow \neg A) \wedge (\neg(A \vee B) \Rightarrow \neg B)) \Rightarrow (\neg(A \vee B) \Rightarrow (\neg A \wedge \neg B))$  [(T0)]
  - 7  $\neg(A \vee B) \Rightarrow (\neg A \wedge \neg B)$  [5, 6, (MP)]
  - 8  $\neg(A \vee B) \equiv (\neg A \wedge \neg B)$  [7, (12b), by def. of  $\equiv$ ]

This completes the proof.

□

This property holds for the constructive negation as well, thus for the systems  $mCZ_n$ . Thus, for these two paraconsistent systems we can define the Kripke semantics in the similar way as for the intuitionistic logic.



## 4 Conclusion

In this paper we have slightly modified a weakening of negation originally presented in the system  $Z_n$  [1] in order to obtain a paraconsistent logic, by eliminating the axiom  $\neg 1 \Rightarrow 0$ . This modified system  $mZ_n$  has a split negation.

Moreover if we preserve also the multiplicative property for this weak split negation we obtain the modified system  $mCZ_n$  with a *constructive* paraconsistent negation which satisfies also the contraposition law for negation.

Both systems have the negation that is different from the (nonparaconsistent) intuitionistic negation (its algebraic counterpart is different from the pseudocomplement of Heyting algebras). In both of them the formula NEFQ is still derivable, but it does not hold the falso quodlibet proof rule. Thus, they satisfy all da Costa conditions (from NdC1 to NdC4).

The Kripke-style semantics for these two paraconsistent negations are defined as *modal* negations: they are a conservative extension of the positive fragment of Kripke semantics for intuitionistic propositional logic [1], where only the satisfaction for negation operator is changed by adopting an incompatibility accessibility relation for this modal operator which comes from Birkhoff polarity theory based on a Galois connection for negation operator.

If we denote by  $Z_n^-$  the system obtained from  $mZ_n$  by eliminating the axiom (12b) (thus with the *general* negation in Definition 2, that is *only antitonic*), then the da Costa axiom (12) can not be derived from the another axioms (but the axiom (11) is still derivable from the antitonic property of the negation). But in such a case, when we really need the da Costa axiom (12), we are not able to define a Kripke-style semantics for this negation operator, based on the Birkhoff polarity. Consequently, this case needs more future investigations.

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